A Sage Library For Analysis Of Nonlinear Binary Mappings

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Outline

Substitutions

Definition

Substitution box (S-box) is an arbitrary mapping of one alphabet to another.

Substitutions for cryptography

S-boxes used in cryptography often map elements from vector space $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$.

Substitutions

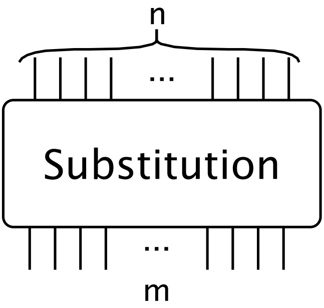


Figure : A Substitution Box

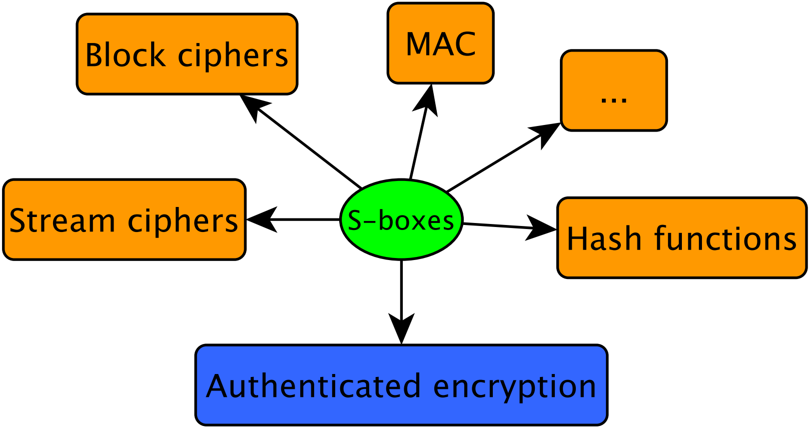
Possible variants

- $n > m$
- $n < m$
- $n = m$
  - $\#img(\text{S-box}) = 2^n$

Representations

- lookup tables
- vectorial Boolean functions
  - Boolean functions
- system of equations

Application of S-boxes



List of properties

Definition

An S-box is a mapping of an n-bit input message to an m-bit output message.

- Minimum degree
- Balancedness
- Nonlinearity
- Correlation immunity
- δ-uniformity
- Cyclic structure
- Algebraic immunity
- Absolute indicator
- Absence of fixed points
- Propagation criterion
- Sum-of-squares indicator
- ...

Cryptographic properties of S-boxes (1/5)

Definition

Let n and m be two positive integers. Any function $F : \mathbb{F}_2^n \mapsto \mathbb{F}_2^m$ is called an (n, m)-function or vectorial Boolean function.

δ-uniform

Arbitrary F is differentially δ-uniform if equation

$$b = F(x) + F(x + a),\ \forall a \in \mathbb{F}_2^n, \forall b \in \mathbb{F}_2^m, a \neq 0$$

has at most δ solutions.

Cryptographic properties of S-boxes (2/5)

Walsh transform

The Walsh transform of an (n, m)-function F at $(u, v) \in \mathbb{F}_2^n \times \mathbb{F}_2^m \backslash \{0\}$

$$\lambda(u, v) = \sum_{x \in \mathbb{F}_2^n} (-1)^{v \cdot F(x) \oplus u \cdot x}, \qquad (1)$$

where "$\cdot$" denotes inner products in $\mathbb{F}_2^n$ and $\mathbb{F}_2^m$ respectively.

Nonlinearity

$$NL(F) = 2^{n-1} - \frac{1}{2} \max_{v \in \mathbb{F}_2^{m*};\ u \in \mathbb{F}_2^n} |\lambda(u, v)|$$

Cryptographic properties of S-boxes (3/5)

Balancedness

An (n, m)-function F is called balanced if it takes every value of F_2^m the same number of times (2^{n-m}).

Absence of Fixed Points

A substitution must not have fixed point, i.e.

$$F(a) \neq a, \quad \forall a \in \mathbb{F}_2^n.$$

Cryptographic properties of S-boxes (4/5)

The algebraic normal form (ANF) of any (n, m)-function F always exists and is unique:

$$F(x) = \sum_{I \subseteq \{1,\dots,n\}} a_I \left(\prod_{i \in I} x_i \right) = \sum_{I \subseteq \{1,\dots,n\}} a_I x^I, \; a_I \in \mathbb{F}_2^m$$

The algebraic degree of F

$$deg(F) = \max\{|I| \mid a_I \neq 0\}$$

Minimum degree

The minimum algebraic degree of all the component functions of F is called the minimum degree.

Cryptographic properties of S-boxes (5/5)

Arbitrary substitution can be represented as the system of equations

$$\begin{cases} g_1(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) = 0; \\ g_2(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) = 0; \\ \ldots \\ g_r(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) = 0. \end{cases} \tag{2}$$

Algebraic immunity

The algebraic immunity $AI(F)$ of any (n, m)-function F is the minimum algebraic degree of all functions in (2).

Outline

System for Algebra and Geometry Experimentation (Sage)

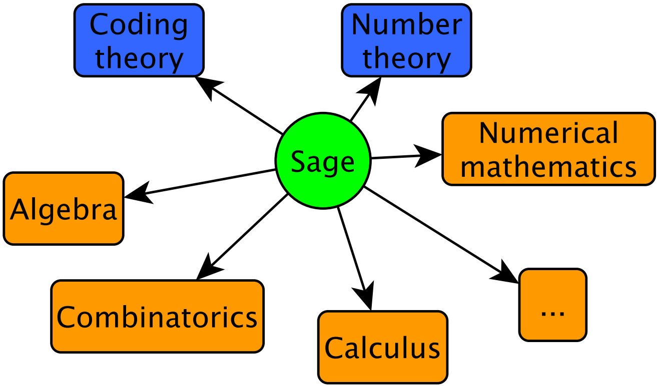


Figure : One can use Sage for ...

System for Algebra and Geometry Experimentation (Sage)

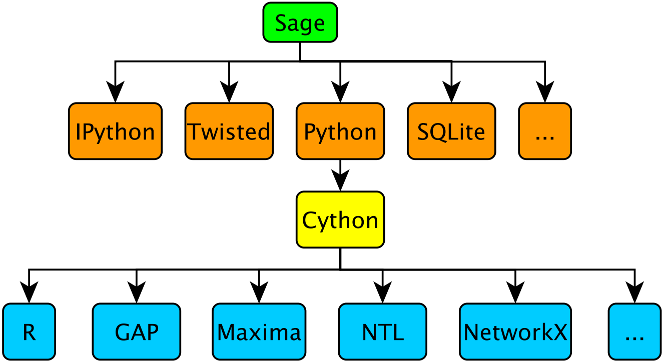
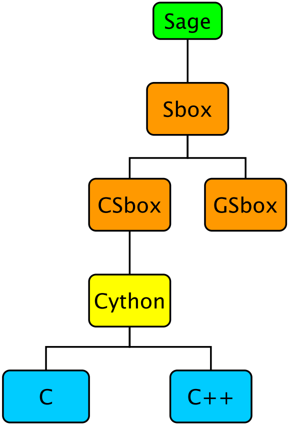


Figure : Sage components

General overview of the Sbox library



Design principles

- Orientation on arbitrary n and m
- Code optimization for performance
- Implementation of known cryptographic indicators

List of supported indicators (CSbox.sage)

- Minimum degree
- Balancedness
- Nonlinearity
- Correlation immunity
- δ-uniformity
- Cyclic structure
- Algebraic immunity
- Absolute indicator
- Absence of fixed points
- Propagation criterion
- Sum-of-squares indicator
- ...

Generation of substitutions (GSbox.sage)

- Gold
- Kasami
- Welch
- Niho
- Inverse
- Dobbertin
- Dicson
- APN for $n = 6$
- Optimal permutation polynomials for $n = 4$
- Polynomial
- ...

Unification of the functions

generate_sbox calls different methods based on parameters method and T which define generation method and equivalence respectively.

Additional functionality

- Extra functions
    - Resilience (balancedness and correlation immunity)
    - Maximum of linear approximation table
    - Check APN property (optimized)
- Convert linear functions to matrices and vice versa
- Apply EA- and CCZ-equivalence
- Generation of substitutions
    - Based on user-defined polynomial (trace supported)
    - Random substitution/permutation
    - With predefined properties
- Input/output
    - Set and get S-boxes as lookup tables
    - Get univariate representation/system of equations
    - Convert polynomial to/from internal representation

Outline

An example

Theorem (Browning, K. A., et al/Budaghyan, L.)

Let α be a multiplicative generator of $\mathbb{F}_{2^6}$ with irreducible polynomial $f(x) = x^6 + x^4 + x^3 + x + 1$. Then the APN function

$$F(x) = \alpha x^3 + \alpha^5 x^{10} + \alpha^4 x^{24}$$

is CCZ-equivalent to an APN permutation over F_{2^6} with $\mathcal{L}(x, y) = (tr_{6/3}(\alpha^4 x) + \alpha tr_{6/3}(y), tr_{6/3}(\alpha x) + \alpha tr_{6/3}(\alpha^4 y))$, where $tr_{6/3} = x + x^{2^3}$, $y = F(x)$.

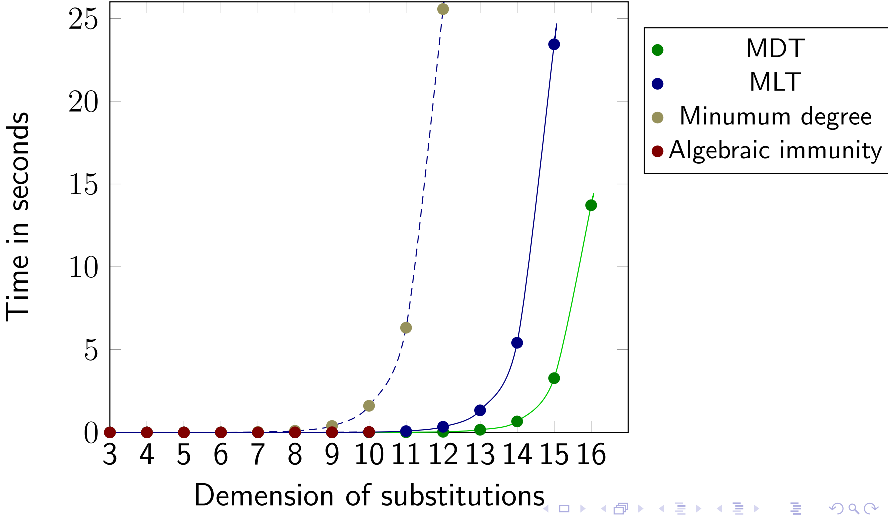
An example

```
sage: %runfile ./Sbox.sage
sage: S = Sbox(n=6,m=6)
sage: P = S.get_ring()
sage: g = S.get_mg()
a
sage: tr = S.Tr_pol(x=P("x"),n=6,m=3)
sage: tr
x^8 + x

sage: M1 = S.l2m(tr.subs(P("(%s)*x"%(g^4))))
sage: M2 = S.l2m(g*tr)
sage: M3 = S.l2m(tr.subs(P("(%s)*x"%(g))))
sage: M4 = S.l2m(g*tr.subs(P("(%s)*x"%(g^4))))
```

An example

```
sage: F = "g*x^3+g^5*x^10+g^4*x^24"
sage: S.generate_sbox(method="polynomial",G=F,T=
    ↪ "CCZ",M1=M1,M2=M2,M3=M3,M4=M4)
sage: S.is_bijection()
True
sage: S.is_APN()
True
sage: S.MDT()
2

sage: S = Sbox(n=6,m=6)
sage: S.generate_sbox(method='APN6')
sage: S.is_bijection()
True
sage: S.is_APN()
True
```

Performance



Comparison of known substitutions

Properties	AES	GOST R 34.11-2012	STB 34.101.31-2011	Kalyna's S-boxes	Next-generation S-boxes
δ-uniformity	4	8	8	8	8
Nonlinearity	112	100	102	96	104
Absolute Indicator	32	96	80	88	80
SSI	133120	258688	232960	244480	194944
Minimum Degree	7	7	6	7	7
Algebraic Immunity	2 (39)	3 (441)	3 (441)	3 (441)	3 (441)

Conclusions

- A high performance library to analyze and generate arbitrary binary nonlinear mappings
- Lots of cryptographic indicators and generation functions are included
- Functionality can be expanded quite easily
- Under development
- Hard to run for the first time
  - Works only in consoles
- Source code: `https://github.com/okazymyrov/sbox`