Homomorphic encryption

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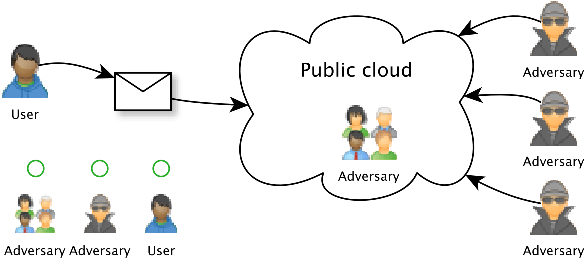
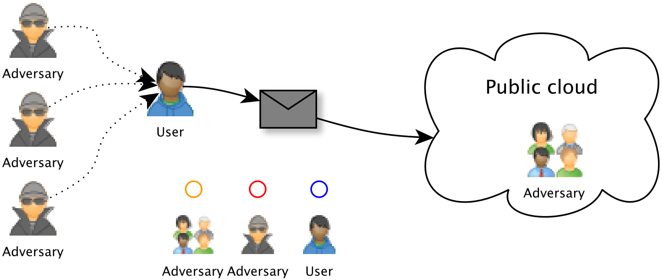
Outline

Cloud computations



Encrypted cloud computations



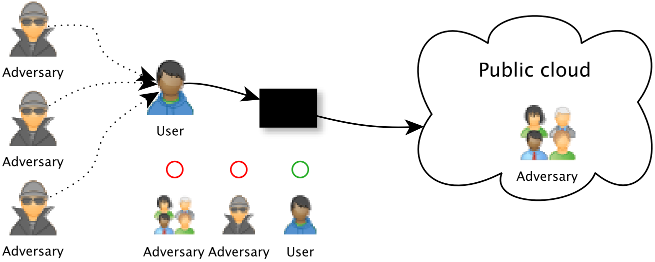
Computing on encrypted data

It would be nice to be able to ...

- encrypt data in the cloud
  - $\hookrightarrow$ while still allowing the cloud to search, sort, edit ...
- keep the data in the cloud in encrypted form
  - $\hookrightarrow$ without need to encrypt/decrypt every time
- encrypt queries to the cloud
  - $\hookrightarrow$ while still allowing the cloud to process them
  - $\hookrightarrow$ the cloud returns encrypted answers

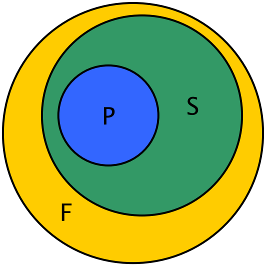
Homomorphic encryption

Cloud computations with homomorphic encryption



What is a homomorphic encryption (HE)?

- An encryption scheme: $(KeyGen, Enc, Dec)$
  - $\star$ $(pk, sk) = KeyGen(rnd),\ c_i = Enc_{pk}(m_i),\ m_i = Dec_{sk}(c_i)$
- A homomorphic encryption scheme: $(KeyGen, Enc, Dec,$ $Eval Eval)$
  - $\star$ $\{c_i'\} = Eval_{pk}(f, \{c_i\})$
  - $\star$ $\{c_i'\} = Eval_{pk}(f, \{c_i\})$, $Dec_{sk}(Eval_{pk}(f, \{Enc_{pk}(x_i)\})) = f(\{x_i\})$



P - partially

S - “somewhat”

F - full

Outline

Partially homomorphic encryption schemes

RSA

KeyGen: $N = p \cdot q$, where p and q large prime numbers

$$gcd(e, \phi(N)) = 1 \Rightarrow d \cdot e \equiv 1 \pmod{\phi(N)}$$

Enc: $c \equiv m^e \pmod{N}$

Dec: $m \equiv c^d \pmod{N}$

$Enc(m_1) \cdot Enc(m_2) = Enc(m_1 \cdot m_2)$:

$$c_1 \equiv m_1^e \pmod{N} \qquad c_2 \equiv m_2^e \pmod{N}$$

$$c_1 \cdot c_2 \equiv m_1^e \cdot m_2^e \equiv (m_1 \cdot m_2)^e \pmod{N}$$

Partially homomorphic encryption schemes

- RSA, ElGamal work for multiplication
- Paillier, Benaloh work for addition
- Goldwasser-Micali works for XOR
- MGH'08 works for degree d polynomials
- ...

$(+, \cdot)$-Homomorphic encryption

It would be really nice to have ...

- Plaintext space $\mathbb{Z}_N$
- Ciphertext space $\mathbb{Z}_N$
- Homomorphic $Enc(x)/Eval(x)$ for both "$+$" and "$\cdot$"

$$Enc(m_1) + Enc(m_2) \equiv Enc(m_1 + m_2 \pmod{N})$$
$$Enc(m_1) \cdot Enc(m_2) \equiv Enc(m_1 \cdot m_2 \pmod{N})$$

- Then we can compute many useful functions on ciphertexts

Outline

Breakthrough

- Genrty'09: a bootstrapping technique
  - ⋆ "Somewhat" homomorphic → Fully homomorphic
- Gentry also described a candidate "bootstrappable" scheme
  - ⋆ Based on ideal lattices
- Gentry's scheme was complex
  - ⋆ it used advanced algebraic number theory
- Can it be simpler?
  - ⋆ polynomials, matrices ... integers

(XOR, AND)-Homomorphic encryption

Why (XOR, AND)?

- because XOR and AND gives a Turing-complete system
  - $\hookrightarrow$ if we can compute XOR and AND on encrypted bits
    - $\hookrightarrow$ we can compute ANY function on encrypted inputs

A secret-key homomorphic encryption

Dijk, Gentry, Halevi and Vaikuntanathan [DGHV'10]

Secret key: large odd number p

Encryption steps of a bit m:

Choose at random large q and small r

$Enc : c = p \cdot q + 2 \cdot r + m$

$\hookrightarrow$ $2 \cdot r + m$ much smaller than p

$\hookrightarrow$ ciphertext is close to a multiple of p

$Dec : m \equiv (c \bmod p) \bmod 2$

Parameters: $|r| = n$, $|p| = n^2$ and $|q| = n^5$

Why is this homomorphic?

$$c_1 = p \cdot q_1 + 2 \cdot r_1 + m_1 \qquad c_2 = p \cdot q_2 + 2 \cdot r_2 + m_2$$

Adding (XORing) two encrypted bits

$$c_1 + c_2 = (q_1 + q_2) \cdot p + 2 \cdot (r_1 + r_2) + (m_1 + m_2)$$

$\hookrightarrow$ if $2 \cdot (r_1 + r_2) + (m_1 + m_2)$ much smaller than p

$\hookrightarrow$ $(c_1 + c_2 \mod p) \mod 2 \equiv m_1 + m_2 \pmod 2$

Multiplying (ANDing) two encrypted bits

$$\begin{aligned} c_1 \cdot c_2 = & q_1 q_2 p^2 + 2 q_1 p r_2 + q_1 m_2 p + 2 q_2 p r_1 + 4 r_1 r_2 + 2_r 1 m_2 + \\ & q_2 m_1 p + 2 m_1 r_2 + m_1 m_2 = \\ & (q_1 q_2 p + 2 q_1 r_2 + q_1 m_2 + 2 q_2 r_1 + q_2 m_1) p + \\ & 2(2 r_1 r_2 + r_1 m_2 + m_1 r_2) + m_1 m_2 \end{aligned}$$

Why is this homomorphic?

$$c_1 = p \cdot q_1 + 2 \cdot r_1 + m_1 \qquad c_2 = p \cdot q_2 + 2 \cdot r_2 + m_2$$

Multiplying (ANDing) two encrypted bits (continue)

$$\begin{aligned}
c_1 \cdot c_2 = \ldots = (q_1 q_2 p + 2q_1 r_2 + q_1 m_2 + 2q_2 r_1 + q_2 m_1)p + \\
2(2r_1 r_2 + r_1 m_2 + m_1 r_2) + m_1 m_2 \\
c_1 q_2 =& q_1 q_2 p + 2q_2 r_1 + q_2 m_1 \\
c_2 q_1 =& q_1 q_2 p + 2q_1 r_2 + q_1 m_2 \\
c_1 \cdot c_2 =& (c_1 q_2 + c_2 q_1 - q_1 q_2 p)p + 2(2r_1 r_2 + r_1 m_2 + m_1 r_2) + m_1 m_2
\end{aligned}$$

$\hookrightarrow$ if $2(2r_1 r_2 + \ldots) + m_1 m_2$ much smaller than p

$\hookrightarrow$ $(c_1 \cdot c_2 \bmod p) \bmod 2 \equiv m_1 \cdot m_2 \pmod 2$

Dijk, Gentry, Halevi and Vaikuntanathan [DGHV'10]

Example

Secret key: choose $p = 99$

Encryption of $m_1 = 1$ and $m_2 = 1$:

Choose $q_1 = 37, r_1 = 3$ and $q_2 = 82, r_2 = 2$

$$c_1 = 99 \cdot 37 + 2 \cdot 3 + 1 = 3670 \quad c_2 = 99 \cdot 82 + 2 \cdot 2 + 1 = 8123$$

Evaluate $m_1 \oplus m_2$ and $m_1 \cdot m_2$:

$$c_1 + c_2 = 11793 \qquad c_1 \cdot c_2 = 29811410$$

Decryption results in:

$$m_1 + m_2 \equiv (11793 \bmod 99) \bmod 2 \equiv 12 \bmod 2 \equiv 0 \pmod 2$$
$$m_1 \cdot m_2 \equiv (29811410 \bmod 99) \bmod 2 \equiv 35 \bmod 2 \equiv 1 \pmod 2$$

Two issues

- Ciphertext grows with each operation

  $\hookrightarrow$ if $|c_1| = |c_2| = n$ then $|c_1 + c_2| = n + 1$ and $|c_1 \cdot c_2| = 2 \cdot n$

  $\hookrightarrow$ we can do lots of additions and some multiplications

  ("somewhat" homomorphic encryption)

- Noise grows with each operation

  $\hookrightarrow$ it doubles on addition and squares on multiplication

  $\hookrightarrow$ after some operations ($|\text{noise}| > \frac{p}{2}$) the ciphertext becomes "unrecoverable"

How secure is this?

Trivial attack

- if there was no noise ($r_1 = 0$ and $r_2 = 0$)
  - $\hookrightarrow$ and encrypted two 0 bits, that is $c_1 = q_1 p$, $c_2 = q_2 p$
    - $\hookrightarrow$ then the secret key $p = GCD(q_1 p, q_2 p)$

Other cases

- there is noise
  - $\hookrightarrow$ the GCD attack doesn't work
    - $\hookrightarrow$ this is called the approximate GCD assumption

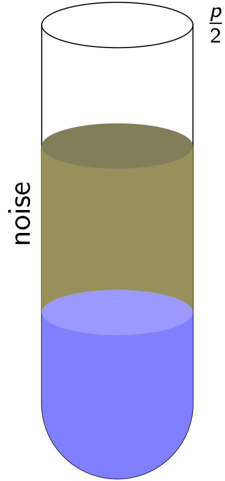
Outline

Gentry's "bootstrapping" theorem

Theorem [Gentry'09]

If an encryption scheme can evaluate its own decryption circuit, then it can evaluate everything.

SHE $\rightarrow$ FHE



- Problem: addition and multiplication increase noise
  - $\hookrightarrow$ addition
  - $\hookrightarrow$ multiplication
- Goal: somehow reduce the noise
- What is the best noise-reduction procedure?
  - $\hookrightarrow$ decryption
    - $\hookrightarrow$ Problem: key is secret
    - $\hookrightarrow$ Goal: reduce noise without publishing secret key

Reduce noise "somewhat" secure

$$Dec_{sk_1}(Eval_{pk_1}(f, \{Enc_{pk_1}(x_i)\})) = f(\{x_i\})$$

$$\begin{aligned}
Dec_{sk_1}(Eval_{pk_1}(Enc_{pk_1}, \{Enc_{pk_1}(x_i)\})) &= Enc_{pk_1}(\{x_i\}) \\
Dec_{sk_1}(Eval_{pk_1}(Dec_{sk_1} \circ Enc_{pk_1}, \{Enc_{pk_1}(x_i)\})) &= \{x_i\} \\
Dec_{sk_1}(Eval_{pk_1}(Enc_{pk_2} \circ Dec_{sk_1} \circ Enc_{pk_1}, \{Enc_{pk_1}(x_i)\})) &= Enc_{pk_2}(\{x_i\})
\end{aligned}$$

$$\begin{aligned}
Enc_{pk_2} \circ Dec_{sk_1} \circ Enc_{pk_1} &= h \\
Dec_{sk_1}(Eval_{pk_1}(h, \{Enc_{pk_1}(x_i)\})) &= Enc_{pk_2}(\{x_i\})
\end{aligned}$$

$$Dec_{sk_2} \circ Enc_{pk_2}(\{x_i\}) = \{x_i\}$$

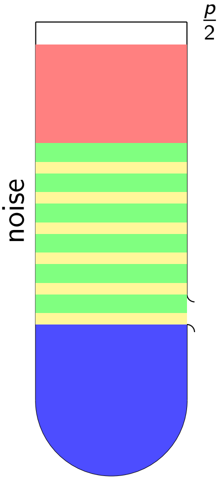
Reduce noise

$$Dec_{sk_1}\left(Eval_{pk_1}\left(f, \{Enc_{pk_1}(x_i)\}\right)\right) = f\left(\{x_i\}\right)$$
$$c_1 = Enc_{pk_1}(m) \quad c_2 = Enc_{pk_1}(pk_2) \quad f = Enc$$

$$c_3 = Eval_{pk_1}\left(Enc, \{c_1, c_2\}\right)$$
$$Dec_{sk_1}(c_3) = Enc_{pk_2}(m)$$

$$c_4 = Enc_{pk_1}(sk_2) \quad f = Dec$$
$$c_5 = Eval_{pk_1}\left(Dec, \{c_3, c_4\}\right)$$
$$Dec_{sk_1}(c_5) = m$$

SHE $\to$ FHE



- Enc_{pk_u} and Dec_{sk_u}
- $(XOR, AND)^t$
- Enc_{pk_i}
- Dec_{sk_i}
- Repeat

Outline

A public-key homomorphic encryption

Dijk, Gentry, Halevi and Vaikuntanathan [DGHV'10]

Secret key: large odd number p

Public key: a set X of large numbers $\{x_1, x_2, .., x_i\}$ such that $x_j \equiv 2 \cdot r_j \pmod{p}$ $x_j \equiv 2 \cdot r_j \pmod{p} \Rightarrow x_j = p \cdot q_j + 2 \cdot r_j$

Encryption steps of a bit m:

Choose at random r and for $Y \subset X$ calculate $b = \sum Y$

$Enc_{pk}(m) : c = b + 2 \cdot r + m$

$Dec_{sk}(c) : m \equiv (c \bmod p) \bmod 2$

$Eval_{pk}(f, \{c_i\})$: as before

Homomorphic properties

$$c_1 = p \cdot \sum_k q_k + 2 \cdot \sum_k r_k + m_1 \qquad c_2 = p \cdot \sum_h q_h + 2 \cdot \sum_h r_h + m_2$$

Addition of two encrypted bits

$$c_1 + c_2 = (\sum_k q_k + \sum_h q_h) \cdot p + 2 \cdot (\sum_k r_k + \sum_h r_h) + (m_1 + m_2)$$

$$m_1 + m_2 \equiv (c_1 + c_2 \mod p) \mod 2$$

Multiplication of two encrypted bits

Assume $Q_1 = \sum_k q_k$, $R_1 = \sum_k r_k$, $Q_2 = \sum_h q_h$, $R_2 = \sum_h r_h$

$$c_1 \cdot c_2 = (c_1 Q_2 + c_2 Q_1 - Q_1 Q_2 p)p + 2(2R_1 R_2 + R_1 m_2 + m_1 R_2) + m_1 m_2$$

$$m_1 \cdot m_2 \equiv (c_1 \cdot c_2 \mod p) \mod 2$$

Example

Secret key: $p = 233$

Public key: $X = \{31955, 36362, 36627, 40098, 45718\}$

Encryption of $m_1 = 0$ and $m_2 = 1$:

Choose $r_1 = 18$, $r_2 = 5$ and $Y = \{31955, 40098\}$

$c_1 = 72053{+}2{\cdot}18{+}0 = 72089$ $c_2 = 72053{+}2{\cdot}5{+}1 = 72064$

Evaluate $m_1 \oplus m_2$ and $m_1 \cdot m_2$:

$$c_1 + c_2 = 144153 \qquad c_1 \cdot c_2 = 5195021696$$

Decryption results in:

$$\begin{aligned} m_1 &\equiv (72089 \bmod 233) \bmod 2 \equiv 92 \bmod 2 \equiv 0 \pmod 2 \\ m_2 &\equiv (72064 \bmod 233) \bmod 2 \equiv 67 \bmod 2 \equiv 1 \pmod 2 \\ m_1 + m_2 &\equiv (144153 \bmod 233) \bmod 2 \equiv 159 \bmod 2 \equiv 1 \pmod 2 \\ m_1 \cdot m_2 &\equiv (5195021696 \bmod 233) \bmod 2 \equiv 106 \bmod 2 \equiv 0 \pmod 2 \end{aligned}$$

Conclusions

- Gentry gave the first feasible result

  $\hookrightarrow$ showing that it can be done "in principle"
- Bootstrapping technique allows transform SHE to FHE

  $\hookrightarrow$ reduce performance dramatically
- New FHEs without bootstrapping

  $\hookrightarrow$ potentially leads to practical implementations
- Lack of independent security checks

  $\hookrightarrow$ public key encryption with homomorphic properties